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A non-linear effect in the capillary instability of liquid jets

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A recent theory for the instability of liquid jets predicted that because of non-linear interactions, a swelling should occur between the crests of the primary disturbance waves. Experiments are described in which such swellings have been observed. The shape of the swellings is in qualitative agreement with the predictions.

1. Introduction

The linearized theory for the stability of a jet of inviscid liquid was first presented by Rayleigh (1945). He showed that the jet is unstable to axisymmetric disturbances provided the wavelength is greater than the circumference. The theory predicts that the unstable disturbances grow exponentially with time, and the maximum growth rate occurs at a wave-number $K = 0{\cdot}697$.

Recently, Yuen (1968) considered disturbances of finite amplitude. In Rayleigh's analysis the volume of the jet is conserved only to the first order in the wave amplitude. By conserving mass to higher orders, Yuen showed that interaction occurred between harmonics of the disturbance, so that energy was extracted from the fundamental. The theory predicts that the neck of the wave diminishes faster than the crest increases. However, at large times the theory also predicts that undulations occur in the trough between crests, that is, the jet exhibits more than one crest per wavelength. Yuen attributed this undulation to a breakdown of the theory, since these secondary waves had not been reported in experimental work.

During an experimental investigation into the distribution of drop sizes produced by the breakup of capillary liquid jets, secondary waves of the type predicted by Yuen were observed, and are here described.

2. Experimental

The experimental method is similar to that of Donnelly & Glaberson (1966); a schematic diagram is given in figure 1. Jets of water were produced from a straight delivery tube of inside diameter 4 mm and length 30 cm. The break-up length of the undisturbed jets was at least 60 cm. The jet velocity past the measuring station was in the range 250–350 cm/s. This is sufficiently high to make

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the effect of acceleration due to gravity small, less than 10 % change in velocity over the jet length, and yet not high enough to cause aerodynamic effects.

To avoid introducing disturbances into the liquid stream from running machinery, and to provide a constant liquid flow-rate for a period of time, an air pressure pumping system was used. By maintaining the pressure over the liquid

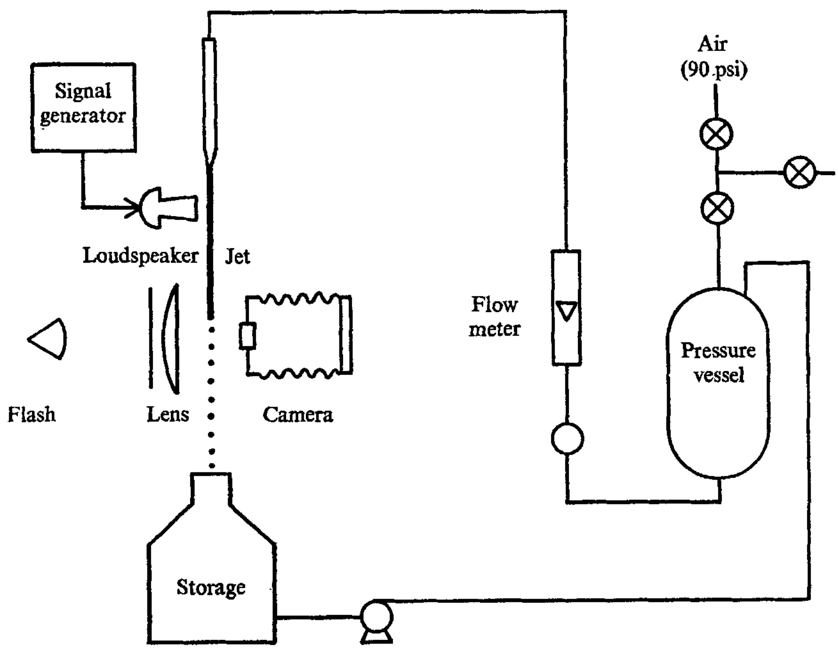

FIGURE 1. Diagrammatic sketch of the apparatus.

in the pressure tank above 40 psig, the pressure change due to the fall in liquid level in the tank during a run of 20 min duration was less than 2 %. A rotameter was used to set up the runs, but the actual flow was measured by catching the liquid from the jet in a graduated cylinder over a known interval of time.

Small perturbations were applied to the jet from a loudspeaker placed about 1 in. away. The speaker was rated at 10 W, and was driven by a Radford 15 W power amplifier. The perturbation frequency was generated by a signal generator (Advance Components Ltd., model J-1) with a frequency range 15 to 50 000 Hz. The amplitude of the perturbation was controlled by varying the power output of the signal generator, and was such that the jet travelled about 20 cm before any visible wave formed, but broke up in a length shorter than the break-up length of the undisturbed jet. The jet could be viewed in the light of a stroboscope, so that experiments could be set up and studied qualitatively before photographs were taken.

Photographs were taken through a Schneider–Kreuznach Repro–Claron 135 mm f 8 lens on a MPP sliding back 4×5 plate camera. The image length on the plate was about 0·6 times actual, so that it was possible to photograph up to 22 cm of jet length. The light source was a Dawe Microflash Junior, which gives 8·5 joules with 10 μsec duration. The light was diffused near the source, further diffused after travelling about 50 cm, and then passed through a 25 cm diameter condensing lens to provide a background of uniform illumination. Exposures were on Ilford FP 4 plates at f 22.

The wave profiles could be measured on a Hilger Universal Measuring Projector. The plate was enlarged fifteen times on a ground glass screen on which it was possible to locate cross-hairs to $\pm 0{\cdot}001$ in. in the axial direction and $\pm 0{\cdot}0001$ in. on the diameter. Full details of the measuring techniques have been given by Rutland (1969).

3. Results and discussion

Three examples of waves growing on water jets are shown in figure 2 (plate 1). The wavelength shown is that of the primary disturbance. This was checked through the relation $\lambda = Q/\pi a^2 f$ where Q is the volumetric liquid flow-rate and f is the frequency of the applied disturbance. In each case there is a secondary swelling between the primary wave crests, as suggested by Yuen's theory. Pronounced secondary waves such as these were very difficult to produce. The primary waves tend to grow exponentially and the growth of the secondary waves is always much slower. The amplitudes of the higher harmonics depend upon powers of the initial amplitude ϵ_0, so that, if one imposes a very small ϵ_0 in order to defer breakup of the jet due to amplification of the fundamental as long as possible, one is at the same time diminishing the amplitude of the secondary wave and making it more difficult to observe.

In determining the shape of the surface of the jet, Yuen assumed that the surface disturbance was of the form

$$\eta(z,t) = \sum_{m=1}^{\infty} \eta_0^m \eta_m, \tag{3.1}$$

where the initial perturbation η_0 can be written as $1+\epsilon_0$, ϵ_0 being the amplitude of the disturbance. Lengths are dimensionless with respect to a, the undisturbed jet radius. The coefficients in the first-order approximation were chosen to make the solution agree with that obtained in the linearized analysis of Rayleigh (1945). The second-order solution contained a purely time-dependent term to offset the addition of volume from the fundamental. The solution, extended to third-order terms, was

$$\begin{aligned}\eta(z,t) = \eta_0 \cos Kz \cosh \omega_1 t + \eta_0^2(B_{22}(t)\cos 2Kz - \tfrac{1}{8}(\cosh \omega_1 t + 1))\\ + \eta_0^3(B_{31}(t)\cos Kz + B_{33}(t)\cos 3Kz),\end{aligned} \tag{3.2}$$

where z is axial distance, K is the wave-number; the time t is dimensionless with respect to $(a^3\rho/T)^{\frac{1}{2}}$. The surface tension is T and the density of the liquid is ρ. The coefficients ω_1, B_{22}, B_{31}, B_{33}, depend on further coefficients a_{11}, etc., all of which are given by Yuen. Typographical omissions were noted in two of these coefficients, which should read

$$b_{22} = \frac{2\omega_1^2 I_b(1-2KI_a) + (2+K^2+\omega_1^2(3-I_a^2))\,K}{4I_b(\omega_2^2-4\omega_1^2)}, \tag{3.3}$$

$$c_{22} = \frac{2+K^2+\omega_1^2(1+I_a^2)}{8(1-4K^2)}. \tag{3.4}$$

In figure 3, profiles calculated according to (3.2) are compared with the measured profile for $K = 0{\cdot}250$. The dimensionless time was calculated by taking the product of the applied frequency and the number of wavelengths between the point of application of the perturbation and the point of observation. However, the initial perturbation ϵ_0 is unknown. The two initial amplitudes were chosen to satisfy different criteria. $\epsilon_0 = 0{\cdot}0008$ gives a wave with maximum amplitude close to that of the experimental wave, and $\epsilon_0 = 0{\cdot}0010$ gives a trough nearly equal in depth to the experimental one. The calculated wave is symmetrical about $Kz/\pi = 1{\cdot}0$, so the predicted profiles indicate four peaks per wavelength, whereas the experiments show only three. In fact, up to the point of jet breakup, we never observed more than one internodal undulation. After breakup, of

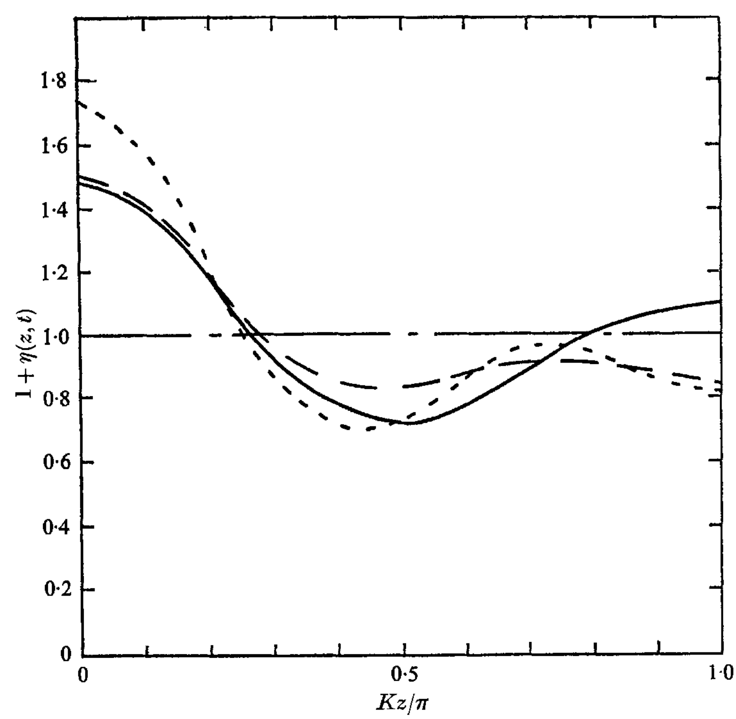

FIGURE 3. Theoretical and experimental profiles of an amplifying disturbance. $K = 0{\cdot}250$; $t = 46{\cdot}7$. ——, experiment; — —, $\epsilon_0 = 0{\cdot}0008$; - - - -, $\epsilon_0 = 0{\cdot}0010$.

course, the ligaments between large drops can often disintegrate further, in which case many undulations per primary wavelength could be observed. It should also be noted that at $t = 46{\cdot}7$ the theory is probably inaccurate anyway. As Yuen pointed out, it was not possible to put a limit on the accuracy of his method without investigating the time behaviour of the higher-order terms. The calculated result in which Yuen found intermediate swellings was for $t = 22{\cdot}0$.

We have assumed that the swellings between the crests of the primary disturbance were harmonics of that disturbance, in the sense that they were composed of waves whose frequencies were integer multiples of the applied frequency, and whose origin lay in the fundamental or in interactions between the fundamental and the various harmonics. An alternative that ought to be explored is that these swellings were caused by higher harmonics in the device producing the primary disturbance. This does not seem likely, because in the range of K which we have examined, the growth rate σ_0 increases with increasing frequency or wave-number. Taking for example $K = 0{\cdot}075$, if the inter-nodal swelling were caused by the loudspeaker, its wave-number would be $K = 0{\cdot}150$, and it would grow approximately as $\exp(2\sigma_0 t)$ rather than as $\exp(\sigma_0 t)$ as with the wave of $K = 0{\cdot}075$. Thus the longer wave would rapidly become swamped, and only the short wave would eventually be seen. In fact this did not occur, and indeed the intermediate waves were generally very difficult to produce.

Although these secondary waves have not been reported before, their existence could have been inferred from the observation that capillary jets do not break up into mono-size droplets, as the linear theory would suggest. Rather, a distribution of drop sizes is produced, some drops coming from the crests of the primary waves and others from the ligament linking them. These so-called 'satellite' droplets are well shown in the photographs of Donnelly & Glaberson (1966), and are a direct result of the intermediate swellings.

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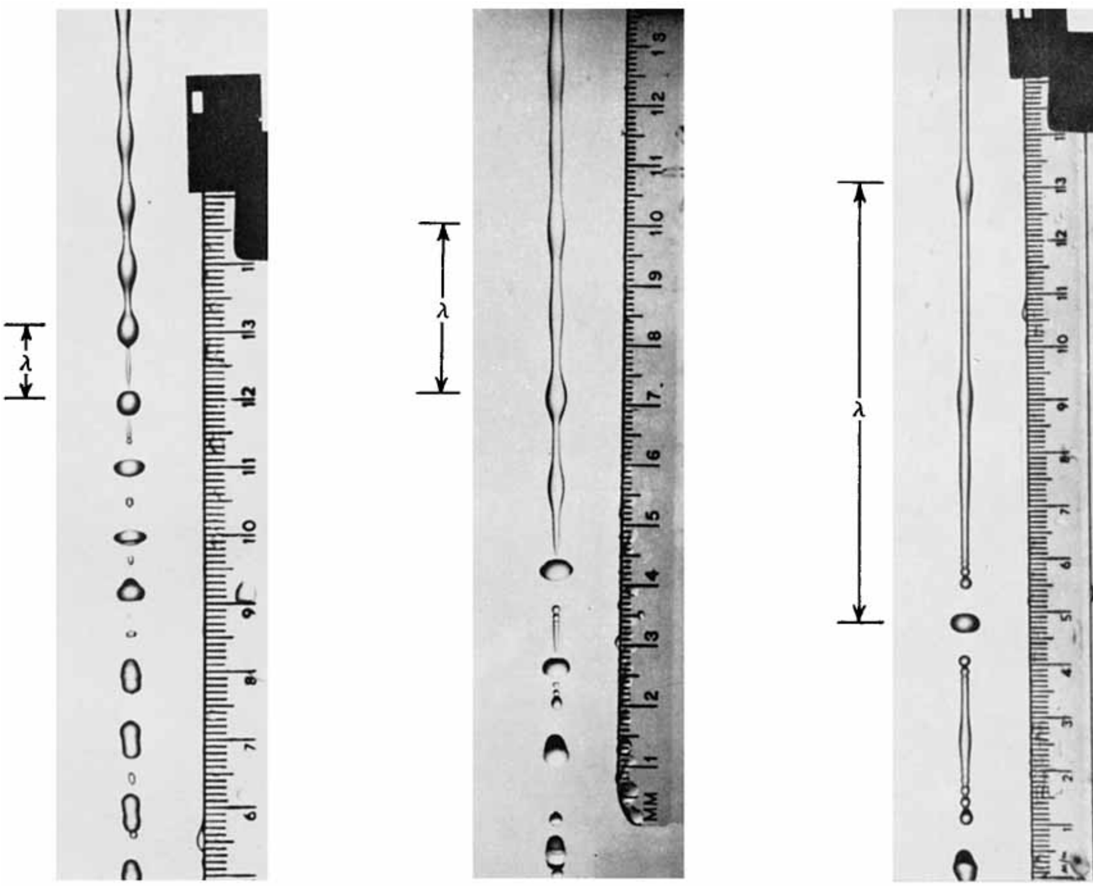

$K = 0{\cdot}683$ $K = 0{\cdot}250$ $K = 0{\cdot}075$

FIGURE 2. Photographs of unstable disturbances at three wave-numbers. Secondary swellings are clearly visible between the wave peaks of the applied perturbation, whose wavelength is λ.